2023 Georgia Tech High School Math Day
Proof Competition

Instructions. Congratulations on advancing to the proof portion of the competition! Do not open this envelope until instructed to do so. Please write your name and school (or N/A) in the designated area at the bottom of this envelope.

This envelope contains the exam questions, answer sheets, and scratch paper. This exam consists of five questions and a tiebreaker question. You will have 120 minutes to complete as much of the exam as possible. Write your initials and the problem number in the designated areas at the top of every answer sheet. Answer sheets without initials and question numbers will not be graded. If you need additional space, include a separate sheet of paper inside your problem folder, labeling it with your initials and the problem number. Indicate page numbers as appropriate. When you are finished, place only the papers you wish to be graded in the envelope and give the envelope to a proctor.

Name/School:
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1. Let $p \in \{2, 3, 5, 7 \dots\}$ be a prime number. Given a rational number x, write $x = p^k \cdot \frac{m}{n}$ where m and n are indivisible by p. We define $v_p(x) = v_p(p^k \frac{m}{n}) = k$. For instance $v_3(\frac{5}{12}) = v_3(3^{-1} \cdot \frac{5}{4}) = -1$. The function v_p is called the p-*adic valuation.*

   Assume that none of $x, y, xy, x + y$ are equal to 0.

   (a) Prove that $v_p(xy) = v_p(x) + v_p(y)$ and $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$.

   (b) Prove that if $v_p(x) \neq v_p(y)$ then $v_p(x + y) = \min\{v_p(x), v_p(y)\}$.

Solution: (a) Write $x = p^i \cdot \frac{s}{t}$ and $y = p^j \cdot \frac{u}{v}$ where s, t, u, v are indivisible by p. Using symmetry, let us assume that $i < j$. First, we have

$$xy = p^{i+j} \cdot \frac{su}{tv}$$

and su, tv are indivisible by p since s, t, u, v were. Therefore $v_p(xy) = i + j = v_p(x) + v_p(y)$.

Second,

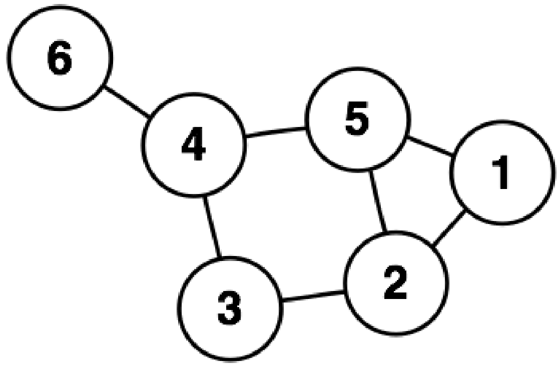
$$x + y = p^i \left(\frac{s}{t} + p^{j-i} uv \right) = p^i \cdot \frac{sv + p^{j-i} tu}{tv}.$$

Since the common denominator is tv, we know that any powers of p which appear in the fraction must do so in the numerator. Meaning it is a non-negative power of p. Therefore $v_p(x + y) \geq i = \min\{v_p(x), v_p(y)\}$.

(b) Using the same analysis, if $j - i > 0$, then we know that $p \nmid sv + p^{j-i}tu$ since $p \mid p^{j-i}tu$ but $p \nmid sv$. Therefore there are no powers of p in the numerator either and so $v_p(x + y)$ must be exactly i.

2. A *graph* G consists of a set of vertices V and a set of edges E, where an edge is an unordered pair of vertices written $e = uv = vu$ with u and v in V.

   For $u \in V$, define the degree of u, written $\deg(u)$, to be the number of edges having u as one of the endpoints. For instance $\deg(1) = 2$ in the above graph because the vertex 1 belongs to 2 edges: $12, 15$.



A graph with $V = \{1, 2, 3, 4, 5, 6\}$ as the set of vertices and
$E = \{12, 15, 23, 25, 34, 45, 46\}$ as the set of edges.

(a) Prove that $\sum_{v \in V} \deg(v) = 2|E|$. In words: the sum of all the degrees of all the vertices is equal to twice the number of edges.

(b) A *tree* is a graph in which there are no cycles (e.g. $1 \to 2 \to 5 \to 1$ is a cycle in the above graph as is $2 \to 3 \to 4 \to 5$) and the graph is "a single piece." Formally, this means that for every pair of vertices $u, v \in V$ there is always a way to travel from u to v by a sequence of edges $u \to v_1 \to v_2 \to \cdots \to v$.

It is known that for trees, the number of edges and number of vertices are related by $|E| = |V| - 1$. Combine this with the previous fact to show that if a tree T has a vertex v with $\deg(v) = k \geq 2$ then T has at least k vertices with degree 1 (called leaf vertices). Hint: let n_i be the number of vertices v with $\deg(v) = i$.

Solution: (a) For every edge $uv \in E$, uv contributes 1 to $\deg(u)$ and 1 to $\deg(v)$. Therefore every edge contributes 2 to the sum of all degrees. Hence the sum must be $2|E|$.

(b) Partition the sum, $\sum_{v\in V} \deg(v)$, according to the degree of v:

$$\begin{aligned}\sum_{v\in V} \deg(v) &= \sum_{i=0}^{\infty} \sum_{v:\deg(v)=i} \deg(v)\\ &= \sum_{i=0}^{\infty} \sum_{v:\deg(v)=i} i\\ &= \sum_{i=0}^{\infty} n_i \cdot i\end{aligned}$$

where $n_i = \#\{v : \deg(v) = i\}$.

On the other hand, this is $2|E| = 2|V| - 2$ (using the fact about trees). Similarly, we can partition $|V| = \sum n_i$. Thus

$$\sum_{i=0}^{\infty} n_i \cdot i = 2\left(\sum_{i=0}^{\infty} n_i\right) - 2.$$

We can now simplify this to

$$\sum_{i=0}^{\infty} n_i(i-2) = -n_1 + 0n_2 + 1n_3 + 2n_4 + \cdots = -2.$$

Now if $n_k \geq 1$, then we have

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k + \cdots \geq 2 + (k-2) = k.$$

3. Let x, y, and z be non-negative real numbers. Prove that

$$(y+z)^3 + 9x^2y + 9x^2z \geq 24xyz.$$

Solution: We have that

$$\begin{aligned}(y+z)^3+9x^2y+9x^2z &= (y+z)\left((y+z)^2+(3x)^2\right)\\ &\geq (y+z)\cdot(2(y+z)(3x))\\ &= 6x(y+z)^2\\ &\geq 24xyz,\end{aligned}$$

and we are done.

4. Find, with proof, all positive integers x, y, and z such that

$$x^y - zy = xy,$$

and z is odd.

Solution: Notice that $x^y = (x+z)y$. Thus, z must have the same prime factors as x. More generally, either $x = kz$ or $z = kx$.

Case 1: $x = kz$. Then, we have $k^y z^y = (k+1)zy$. Since $(k+1) \nmid k$, $k+1 \mid z^y$. Since z is odd, $k+1$ is odd and so k is even. This means that 2^y divides the left-hand side, so 2^y divides the right hand side and thus must divide y. However, $2^y > y$ so this is a contradiction.

Case 2: $kx = z$. Then, we have $x^y = (k+1)xy$. Notice that this means x is odd, since $x \mid z$. k must also be odd, since $k \mid z$. However, this yields an obvious contradiction as $k+1$ must be even so the right-hand side is even while the left-hand side is odd.

Since in either case there is a contradiction, z cannot be an odd number. Hence, there are no positive integers that satisfy the conditions.

5. Given a triangle ABC, with $AB \neq AC$, let D lie on side $\overline{BC}$ such that $\overline{AD}$ bisects $\angle BAC$. Let ℓ be a line tangent to the circumcircles of $\triangle ABD$ and $\triangle ACD$. Prove that ℓ and the perpendicular bisector of segment $\overline{AD}$ meet on side $\overline{BC}$.

Solution: Henceforth, we denote the circumcircle of a triangle $\mathcal{T}$ by $(\mathcal{T})$.

Since PQ is tangent to (ABD), $\angle APQ = \angle ABP = \alpha$, and since PQ is tangent to (ACD), $\angle AQP = \angle ACQ = \beta$. Let lines BP and CQ intersect at a point R. Note that $\angle APR = \angle ADB = \angle AQC$, which means that quadrilateral $APRQ$ is cyclic. Hence, $\angle BRC + \angle PAQ = 180°$, so $\angle BRC = \alpha + \beta$.

Note that

$$\angle BRC = 180° - \angle RBC - \angle RCB = 180° - \alpha - \angle B - \beta - \angle C = \angle A - \alpha - \beta,$$

so $\alpha + \beta = \angle A - \alpha - \beta$, which implies that $\alpha + \beta = \angle A/2$.

By angle chasing, we obtain that

$$\angle BPQ + \angle QCB = \angle BPA + \alpha + \beta + \angle C = \angle B + \angle C + \angle A/2 + \alpha + \beta = 180°,$$

so $BPQC$ is cyclic as well.

In addition,

$$\begin{aligned}\angle PAB &= 180° - \angle BPA - \alpha \\ &= 180° - \angle B - \angle A/2 - \alpha \\ &= \angle A/2 + \angle C - \alpha \\ &= \beta + \angle C = \angle PQA + \angle ACB,\end{aligned}$$

which implies that (APQ) and (ABC) are tangent at A. (will rewrite and finish later)

6. (Tiebreaker) Suppose that the roots of the quadratic $x^2 + bx + c$ are α and β. That is, suppose $x^2 + bx + c = (x - \alpha)(x - \beta)$. Show that $(\alpha - \beta)^2 = b^2 - 4c$.

   Next, suppose that the roots of a cubic $f = x^3 + bx^2 + cx + d$, where b, c, d are real numbers, are α, β, γ. Define $\Delta = (\alpha - \beta)^2(\alpha - \gamma)^2(\beta - \gamma)^2$. Show that

   1. $\Delta = 0$ if and only if f has a repeated root
   2. $\Delta < 0$ if and only if f has 1 real root and a pair of complex roots
   3. $\Delta > 0$ if and only if f has 3 distinct real roots

Solution: If we expand $(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$ we find that $\alpha + \beta = -b$ and $\alpha\beta = c$. Therefore

$$(\alpha - \beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = b^2 - 4c.$$

1. Using the zero-product theorem: if a product of terms is 0 at least one term must be 0
2. If $\alpha = w + zi$ and $\beta = w - zi$ then $(\alpha - \beta)^2 = (2zi)^2 < 0$. The terms $(\alpha-\gamma)^2$ and $(\beta-\gamma)^2$ are conjugate and a product of conjugate complex numbers is positive. Therefore the whole product must be negative.

   Conversely, if $\Delta < 0$ then α, β, γ cannot be all real numbers or else a product of squares would be non-negative.
3. Follows from 1. and 2. because this is the last remaining case.