Georgia Tech High School Math Day

Algebra Competition

April 22, 2023

- Each correct answer is worth one point; there is no deduction for incorrect answers.
- Make sure to write your name and school (or N/A) clearly on the answer sheet
- You may use the test booklet as scratch paper, but no credit will be given for work in the booklet.
- You may keep the test booklet after the test has ended.

Problems contributed by Taiki Aiba, Wade Bloomquist, Trevor Gunn, Michael Lacey, and Hugo Zhou

1. What is the smallest positive integer n such that $\sqrt[4]{n+2023}$ is an integer?

> **Solution:** 378
>
> Since $\sqrt[4]{n+2023}$ is greater than $6^4 = 1296$, it must be $7^4 = 2401$. Thus, our answer is $n = 2401 - 2023 = 378$.

2. A circle in the coordinate plane has its center at $(0,0)$ and passes through the points $(3,a)$ and $(a-2,4)$. What is a?

> **Solution:** $\frac{11}{4}$
>
> Recall that the equation of a circle with center $(0,0)$ is $x^2+y^2=r^2$. Thus, we get that $3^2+a^2=(a-2)^2+4^2$, so $a=\frac{11}{4}$.

3. The areas of three triangles form an arithmetic progression. If one of the triangles has side lengths 8, 15, and 17, and another one of the triangles has side lengths 7, 24, and 25, what is the sum of the possible areas of the third triangle?

> **Solution:** 216
>
> The areas of the two triangles are 60 and 84. Hence, the possible areas of the third triangle are
>
> $84, 60, \mathbf{36} \quad \& \quad \mathbf{36}, 60, 84$
> $60, \mathbf{72}, 84$
> $60, 84, \mathbf{108} \quad \& \quad \mathbf{108}, 84, 60$
>
> The requested sum is equal to $36+72+108=216$.

4. Suppose that a rectangle $ABCD$ satisfies $AB : AC = 8 : 17$. If the area of $ABCD$ is $\frac{24}{5}$, what is the length of $\overline{BC}$?

> **Solution:** 3
>
> Let $AB = 8x$ and $AC = 17x$. Then, by the Pythagorean Theorem,
>
> $$(AB)^2 + (BC)^2 = (AC)^2 \quad \Longrightarrow \quad BC = 15x.$$
>
> Thus, $8x \cdot 15x = \frac{24}{5} \implies x^2 = \frac{24}{600}$, so $x = \frac{1}{5}$ and $BC = 3$.

5. A high school math team has 10 different students and 15 identical cookies to hand out. If each student receives at least one cookie, and no student gets more than two cookies, in how many ways can the cookies be distributed?

> **Solution:** 252
>
> We first give each student 1 cookie, and then since no student can get more than 2 cookies, we figure out how we can give 5 identical cookies to 10 different students such that each student gets at most 1 of these cookies. This gives us $\binom{10}{5} = 252$ ways.

6. A farmer must bring water to her sick cow. The farmer is at coordinates $(0, 12)$ and the cow is at coordinates $(10, 8)$. The river where she will fill her bucket from is the line $y = 0$. At which point $(x, 0)$ along the river should the farmer walk towards to yield the shortest round trip to the cow?

> **Solution:** $x = 6$
>
> The trick is to imagine the farmer crossing through the river to a phantom point at $(10, -8)$. The shortest path is given by a straight line from $(0, 12)$ to $(10, -8)$ which crosses the line $y = 0$ at $x = \frac{12}{12+8} \cdot 10 = 6$. (Draw a picture!)

7. Let p and q be the roots of $x^2 - 6x + 4$ with $p > q$. What is $p^2 - q^2$?

> **Solution:** $12\sqrt{5}$
>
> Recall that $p^2 - q^2 = (p+q)(p-q)$. By Vieta's formulas, we get that $p+q = -(-6) = 6$, and by the discriminant of $x^2 - 6x + 4$, we get that $p - q = \sqrt{(-6)^2 - 4 \cdot 1 \cdot 4} = 2\sqrt{5}$. Thus, our answer is $6 \cdot 2\sqrt{5} = 12\sqrt{5}$.

8. Suppose a and b are irrational numbers such that $a^3 \approx 31.00627$ and $b^3 \approx 20.08554$. What is

$$(a - b)\left((a + b)^2 - ab\right),$$

rounded to the nearest hundredth?

Solution: 10.92

The given expression is equal to

$$(a-b)((a+b)^2-ab) = (a-b)(a^2+b^2+ab) = a^3-b^3 \approx 31.00627-20.08554 \approx 10.92.$$

9. A 60 person party wants to order pizza. There are 40 people who want pepperoni, 16 who want cheese, and 50 who want supreme. A total of 40 people want at least two kinds of pizza and 12 want all three kinds. How many people don't want any pizza?

Solution: This is a direct application of the inclusion-exclusion principle.

The number of people who want pizza is

$$40+16+50-(40+2*12)+12 = 54,$$

so 6 people do not want pizza.

10. Given that $f(x) = 4x^3-6x^2+4x-1$, what is

$$f(2)+f(3)+\cdots+f(100)?$$

Solution: 99999999

Note that $f(x) = x^4-(x-1)^4$, so our answer is

$$(2^4-1^4)+(3^4-2^4)+(4^4-3^4)+\cdots+(100^4-99^4) = 100^4-1^4 = 99999999.$$

11. The area of a right triangle $\triangle ABC$ is X and the area of the inscribed circle of $\triangle ABC$ is Y. If $X^2 = 100Y$, what is the circumference of $\triangle ABC$?

Solution: $20\sqrt{\pi}$

Suppose the inscribed circle of $\triangle ABC$ separates the hypotenuse into two parts with length a and c, and separates the two other sides into parts with length a, b and b, c

respectively. Pythagorean theorem gives

$$\begin{aligned} & (a+b)^2 + (b+c)^2 = (a+c)^2 \\ \implies & b(a+b+c) = ac \\ \implies & C_{\triangle ABC} = \frac{2ac}{b} \end{aligned}$$

where $C_{\triangle ABC}$ is the circumference. On the other hand, the area of $\triangle ABC$ is

$$\frac{1}{2}(a+b)\cdot(b+c) = \frac{1}{2}(b(a+b+c)+ac) = ac$$

and the area of the inscribed circle is πb^2. Plug in the value $C_{\triangle ABC} = 2\sqrt{\pi}X/\sqrt{Y} = 20\sqrt{\pi}$.

12. If the side lengths of a triangle are 5, 13, and 17, compute the largest possible value of $\cos\theta$, where θ is one of the three interior angles of the triangle.

Solution: $\frac{433}{442}$

Let the triangle have vertices A, B, and C, and let $AB = 5$, $AC = 13$, and $BC = 17$. It is well-known that in a triangle, the interior angle opposite the shortest side will have the smallest measure, or in this case, the greatest cosine. Thus, we want to find $\cos(\angle ACB)$ since it is opposite side $\overline{AB}$, which is the shortest side. Law of Cosines gives us

$$\cos(\angle ACB) = \frac{AC^2 + BC^2 - AB^2}{2\cdot AC\cdot BC} = \frac{13^2+17^2-5^2}{2\cdot 13\cdot 17} = \frac{433}{442}.$$

13. Katherine flips an odd coin repeatedly, which is defined on a sequence of flips as follows:

- the probability that the coin will flip heads on flip n is denoted as a_n,
- $a_1 = \frac{1}{2}$, and
- for all $n \geq 1$, if m of the first n flips are tails, a_{n+1} is $\frac{m}{n}$.

What is the probability that $a_6 = \frac{4}{5}$?

Solution: $\frac{1}{24}$

If $a_6 = \frac{4}{5}$, then 4 of the first 5 flips are tails. The first two flips are always either HT or TH, so we must flip TTT afterwards, which has probability $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}$.

14. Triangle ABC has $AB = AC = 17$ and points E and F on sides $\overline{BC}$ and $\overline{AB}$, respectively, with $BE = CE = 8$. Line segments $\overline{AE}$ and $\overline{CF}$ intersect at a point D. If $CD = 10$, what is the length of $\overline{DF}$?

Solution: $\frac{30}{7}$

By the Pythagorean Theorem, we have that $DE = 6$, meaning that $AD = 9$. We then proceed with mass points. Assign a mass of 3 to point B. Then, since $BE = CE$, we have a mass of 3 at point C as well. Then, we have that the mass at point E is the sum of the masses of points B and C, or $3 + 3 = 6$. Since $AD : DE = 3 : 2$, we have that the mass at point A is $\frac{2}{3} \cdot 6 = 4$. Then, the mass at point F is the sum of the masses of points A and B, or $4 + 3 = 7$. Thus, we have that the masses of points C and F are 3 and 7, respectively, so $CD : DF = 7 : 3$. Since we are given that $CD = 10$, we get that $DF = \frac{3}{7} \cdot 10 = \frac{30}{7}$.

15. The polynomials $p(x) = ax^3 + bx^2 + cx + d$ and $q(x) = ex + f$ are written on a whiteboard, where the first polynomial is directly to the left of the second polynomial. A student chooses, randomly and without replacement, three coefficients from the set $\{a, b, c, d, e, f\}$. He then erases those three coefficients from the whiteboard, while keeping the rest intact. Next, he fills in the three missing coefficients with a 1, a 2, and a 3, going from left to right. After that, the not-chosen coefficients in the set $\{a, b, c, d, e, f\}$ are all replaced with 0s on the whiteboard. In the resulting polynomials, what is the expected value of the product $p(1) \cdot q(1)$?

Solution: $\frac{32}{5}$

First, we note that $p(1)$ and $q(1)$ are the sum of the coefficients of p and q respectively.

Case 1: d and e are both replaced with 0s, which happens when three of a, b, c, d are chosen, which means that $p(1) = 1 + 2 + 3 = 6$ and $q(1) = 0$. Then, the value of $p(1) \cdot q(1)$ is 0. This case has probability $\frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}$ of happening, so this case contributes $\frac{1}{5} \cdot 0 = 0$ to our expected value.

Case 2: One of d and e are chosen, but not the other. Then, two of a, b, c, d are chosen, which means that $p(1) = 1 + 2 = 3$, and $q(1) = 3$. Then, the value of $p(1) \cdot q(1)$ is 9. This case has probability $\frac{2\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5}$ of happening, so this case contributes $\frac{3}{5} \cdot 9 = \frac{27}{5}$ to our expected value.

Case 3: Both of d and e are chosen. Then, one of a, b, c, d are chosen, which means that $p(1) = 1$ and $q(1) = 2 + 3 = 5$. Then, the value of $p(1) \cdot q(1)$ is 5. This case has probability $\frac{\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5}$ of happening, so this case contributes $\frac{1}{5} \cdot 5 = 1$ to our expected value.

Adding these cases up, we get a total of $0 + \frac{27}{5} + 1 = \frac{32}{5}$.

16. What is

$$\sum_{d \text{ divisor of } 2023} d^2?$$

In other words, what is the sum of all possible d^2, where d is a divisor of 2023?

Solution: 4190550

The prime factorization of 2023 is $7 * 17^2$. With this, we can deduce that the divisors of 2023 are $1, 17, 17^2, 7, 7 * 17, 7 * 17^2$ and their squared values are $1, 17^2, 17^4, 7^2, 7^2 * 17^2, 7^2 * 17^4$, and their sum is $1 + 17^2 + 17^4 + 7^2 + 7^2 * 17^2 + 7^2 * 17^4$. We can factor out $(1 + 17^2 + 17^4)$ from the above and rewrite our sum as $(1 + 17^2 + 17^4)(1 + 7^2)$. $(1+7^2) = 50$, while $(1+17^2+17^4) = (17^2+1)^2 - 17^2 = (17^2+1-17)(17^2+1+17) = 273 * 307 = 83811$. Multiplying $1 + 7^2 = 50$ by $1 + 17^2 + 17^4 = 83811$ yields our desired answer of 4340550

17. Let x, y, z be complex numbers such that

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 1, \\ x^3 + y^3 + z^3 &= 6. \end{aligned}$$

What is xyz?

Solution: $xyz = 5$

Let $p_i = x^i + y^i + z^i$ and let $e_1 = x+y+z, e_2 = xy+xz+yz, e_3 = xyz$. By Newton's identities, we have

$$p_1 - e_1 = 0$$
$$p_2 - e_1p_1 + 2e_2 = 0$$
$$p_3 - e_1p_2 + e_2p_1 - 3e_3 = 0$$

Solving this gives $e_1 = 3, e_2 = 4, e_3 = 5$.

18. For which integer $n \in \{11, 12, 13, 14\}$ is $\binom{2n}{n} = \frac{(2n)\cdot(2n-1)\cdot(2n-2)\cdots(n+1)}{n\cdot(n-1)\cdot(n-2)\cdots 1}$ divisible by 60?

Solution: The answer is 14. $60 = 3 \cdot 4 \cdot 5$. And Kummer's Theorem says that $\binom{2n}{n}$ is divisible by prime p^k, for prime p if adding n to itself base p, there are at least k carries. So, the question is for which integer above, writing it base $p = 3, 5$, there is a digit greater or equal to $p/2$ in each base. And, base 2, are there at least two non-zero digits. Writing things first in base 3,

$$10 \equiv 101$$
$$11 \equiv 102$$
$$12 \equiv 110$$
$$13 \equiv 111$$
$$14 \equiv 112$$

one sees that 11 or 14 are the only possibilities. Base 5, we have $11 \equiv 21$, so it can't be the answer. Turning to 14, we have

$$14 \equiv 1110(\text{base } 2) \equiv 24(\text{base } 5)$$

So, 14 is the answer. (Inspired by an old conjecture of Erdos and Graham, and a recent paper of Ernie Croot, Hamed Mousavi and Maxie Schmidt. If you don't do Kummer, it can be done, but would take some effort. Kummer's theorem says more, and you can make more elementary questions based on it.)

19. What is the sum of all positive integers n such that the number $57^n + 64^n$ is one more than a multiple of $n!$? (Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.)

Solution: 15

Note that $57^n + 64^n \equiv 1^n + 1^n \equiv 2 \pmod 7$, so all $n \geq 7$ do not work. Thus, we are limited to $n \leq 6$. Next, note that $57^n + 64^n \equiv 2^n + (-1)^n \pmod 5$. When $n = 6$, then $2^6 + (-1)^6 \equiv 3 \pmod 5$, which makes $n = 6$ not work. However, when $n = 5$, we have that $2^5 + (-1)^5 \equiv 1 \pmod 5$. Since $5! = 2^3 \cdot 3 \cdot 5$, we only need to test modulo 3 and modulo 8. We see that $57^n + 64^n \equiv 1 \pmod 3$ for all n, and $57^n + 64^n \equiv 1 \pmod 8$ for all n. Thus, we have that all n between 1 and 5, inclusive, work, for an answer of $1 + 2 + 3 + 4 + 5 = 15$.

20. Two rays emanating from a point form a $60°$ angle. Circles ω_1 and ω_2 of radius 4 are such that ω_1 is tangent to both rays, and ω_2 is tangent to only one of the rays, on the same side of the ray as ω_1. If there exists a circle of radius 3, externally tangent to ω_2 and tangent to both rays, what is the distance between the centers of ω_1 and ω_2?

Solution: $3\sqrt{3}$

Let the circle with radius 3 be ω_3. Let the centers of ω_1, ω_2, and ω_3 be O_1, O_2, and O_3, respectively. Let P be the point from which the two rays emanate. Ignore ω_1 entirely and only focus on ω_2 and ω_3. Now, drop perpendiculars from O_3 and O_2 to the same ray and call the feet Q and R, respectively. We see that $O_2O_3 = 7$ because we are adding the radii of ω_2 and ω_3 since they are externally tangent. We use the Pythagorean Theorem to find that

$$QR = \sqrt{7^2 - (4-3)^2} = 4\sqrt{3}.$$

We can also find the length PQ by seeing that $\triangle O_3QP$ is a $30-60-90$ right triangle. Since $O_3Q = 3$, it follows that $PQ = 3\sqrt{3}$. Therefore, $PR = 3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}$. Now, we add ω_1 back in. Drop a perpendicular from O_1 to the same ray as Q and R and call the foot S. We see that $\triangle O_1SP$ is a $30 - 60 - 90$ right triangle. Since $O_1S = 4$, it follows that $PS = 4\sqrt{3}$. Finally, we are looking for SR because $\overline{SR} \parallel \overline{O_1O_2} \implies SR = O_1O_2$. Thus, our answer is

$$O_1O_2 = SR = PR - PS = 7\sqrt{3} - 4\sqrt{3} = 3\sqrt{3}.$$